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**DIFFERENTIAL AND INTEGRAL CALCULUS FOR  
FUNCTIONS WITH SEVERAL VARIABLES**

The Educational Textbook  
for Students of Technical Universities

**ДИФЕРЕНЦІАЛЬНЕ ТА ІНТЕГРАЛЬНЕ ЧИСЛЕННЯ  
ФУНКЦІЙ БАГАТЬОХ ЗМІННИХ**

Навчальний посібник  
для студентів технічних спеціальностей

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For students and teachers of technical specialties.

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